# THE $L^P$ BOUNDEDNESS OF WAVE OPERATORS FOR SCHRÖDINGER OPERATORS WITH THRESHOLD SINGULARITIES

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Abstract. Let $H = -\Delta + V$ be a Schrödinger operator on $L^2(\mathbb{R}^n)$ with real-valued potential $V$ for $n > 4$ and let $H_0 = -\Delta$. If $V$ decays sufficiently, the wave operators $W_\pm = s - \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}$ are known to be bounded on $L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$ if zero is not an eigenvalue, and on $1 < p < \frac{n}{2}$ if zero is an eigenvalue. We show that these wave operators are also bounded on $L^1(\mathbb{R}^n)$ by direct examination of the integral kernel of the leading term. Furthermore, if $\int_{\mathbb{R}^n} V(x)\phi(x)\,dx = 0$ for all eigenfunctions $\phi$, then the wave operators are $L^p$ bounded for $1 \leq p < n$. If, in addition $\int_{\mathbb{R}^n} xV(x)\phi(x)\,dx = 0$, then the wave operators are bounded for $1 \leq p < \infty$.

## 1. Introduction

Let $H = -\Delta + V$ be a Schrödinger operator with potential $V$ and $H_0 = -\Delta$. If $V$ is real-valued and satisfies $|V(x)| \lesssim \langle x\rangle^{-2-}$, then it is well known that the spectrum of $H$ is the absolutely continuous spectrum on $[0,\infty)$ and a finite collection of non-positive eigenvalues, [19]. The wave operators are defined by the strong limits on $L^2(\mathbb{R}^n)$

$$W_\pm = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}. \tag{1}$$

Such limits are known to exist and are asymptotically complete for a wide class of potentials $V$. That is, the image of $W_\pm$ is equal to the absolutely continuous subspace of $L^2(\mathbb{R}^n)$ associated to the Schrödinger operator $H$. Furthermore, one has the identities

$$W_\pm^* W_\pm = I, \qquad W_\pm W_\pm^* = P_{ac}(H), \tag{2}$$

with $P_{ac}(H)$ the projection onto the absolutely continous spectral subspace associated with the Schrödinger operator $H$.

We say that zero energy is regular if there are no zero energy eigenvalues or resonances. There is a zero energy eigenvalue if there is a solution to $H\psi = 0$ with $\psi \in L^2(\mathbb{R}^n)$, and a

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resonance if $\psi \notin L^2(\mathbb{R}^n)$ is in an appropriate space which depends on the dimension. We note that resonances only occur in dimensions $n \leq 4$. There is a long history of results on the existence and boundedness of the wave operators. We note that Yajima has established $L^p$ and $W^{k,p}$ boundedness of the wave operators for the full range of $1 \leq p \leq \infty$ in [21, 22, 23] in all dimensions $n \geq 3$, provided that zero energy is regular under varying assumptions on the potential $V$. The sharpest result in $n = 3$ was obtained by Beceanu in [3].

If zero is not regular, in general the range of $p$ on which the wave operators are bounded shrinks. Yajima (for $n$ odd), and Yajima and Finco (for $n$ even) proved in [25, 9] that for each $n > 4$, the wave operators are bounded on $L^p(\mathbb{R}^n)$ when $\frac{n}{n-2} < p < \frac{n}{2}$, and on $\frac{3}{2} < p < 3$ in $n = 3$ if zero is not regular. In [16] Jensen and Yajima showed that the wave operators are bounded if $\frac{4}{3} < p < 4$ when $n = 4$ when there is an eigenvalue but no resonance at zero. D'Ancona and Fanelli in [4] show that the wave operators are bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$ in the case of a zero energy resonance, which had roots in the work of Weder, [20]. To the best of the authors' knowledge, there are no results in the literature when zero is not regular and $n = 2$. Very recently Yajima, in [26], reduced the lower bound on $p$ to $1 < p < \frac{n}{2}$ for dimensions $n > 4$ when there is a zero energy eigenvalue. We extend this result to include the $p = 1$ endpoint.

One important property of the wave operators is the intertwining identity,

$$f(H)P_{ac} = W_{\pm} f(-\Delta) W_{\pm}^*,$$

which is valid for Borel functions $f$. This allows one to deduce properties of the operator $f(H)$ from the much simpler operator $f(-\Delta)$, provided one has control on mapping properties of the wave operators $W_{\pm}$ and $W_{\pm}^*$. In dimensions $n \geq 5$, boundedness of the wave operators on for the range of $p$ proven in [25, 9] imply the dispersive estimates

$$\|e^{itH} P_{ac}(H)\|_{L^p \to L^{p'}} \lesssim |t|^{-\frac{n}{2}+\frac{n}{p}}.$$

Here $p'$ is the conjugate exponent satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. In this way, one can use the $L^p$ boundedness of the wave operators to deduce dispersive estimates for the Schrödinger evolution. There has been much work on dispersive estimates for the Schrödinger evolution with zero energy obstructions in recent years by Erdoğan, Schlag and the authors in various combinations, see [8, 10, 7, 5, 11, 12] in which $L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ were studied for all $n > 1$. This work has roots in previous work of [17] and [15, 18] in which the dispersive estimates were studied as operators on weighted $L^2(\mathbb{R}^n)$ spaces.

The range of $p$ proven in [25] allows one to deduce a decay rate of size $|t|^{-\frac{n}{2}+2+}$. This paper is motivated by the recent work of the authors, [11, 12], in which dispersive estimates with a decay rate of $|t|^{2-\frac{n}{2}}$ were proven in the case of an eigenvalue at zero energy, and faster decay if the zero energy eigenspace satisfies certain cancellation conditions. Let $P_e$ be the projection onto the zero energy eigenspace, and write $P_eV1 = 0$ if $\int_{\mathbb{R}^n} V(x)\phi(x)\,dx = 0$ for each eigenfunction $\phi$, and $P_eVx = 0$ if $\int_{\mathbb{R}^n} xV(x)\phi(x)\,dx = 0$. Considering the linear Schrödinger evolution as an operator from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, time decay of size $|t|^{1-\frac{n}{2}}$ is observed if $P_eV1 = 0$ and if in addition $P_eVx = 0$, the decay rate improves to $|t|^{-\frac{n}{2}}$.

Time decay of these orders would be consistent with $L^p$ boundedness of the wave operators over the range $1 \le p \le n$ if $P_eV1 = 0$. In the case that $P_eVx = 0$ as well, the time-decay is identical to what occurs in the free case, so it is conceivable for the range to extend to $1 \le p \le \infty$. Our main result confirms this to be the case for all $p$ except the upper endpoints. During the review period for this article Yajima additionally showed that the orthogonality conditions are also necessary for the extended range of $L^p$ boundedness, [27].

**Theorem 1.1.** *For each $n > 4$, let $n_* = \frac{n-1}{n-2}$. Assume that $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for $\beta > n+3$, or $\beta > 16$ if $n = 6$, and*

$$\mathcal{F}\big(\langle\cdot\rangle^{2\sigma}V\big) \in L^{n_*}(\mathbb{R}^n) \text{ for some } \sigma > \frac{1}{n_*}. \tag{3}$$

*i) The wave operators extend to bounded operators on $L^p(\mathbb{R}^n)$ for all $1 \le p < \frac{n}{2}$.*

*ii) If $\int_{\mathbb{R}^n} V(x)\phi(x)\,dx = 0$ for all zero-energy eigenfunctions $\phi$, then the wave operators extend to bounded operators on $L^p(\mathbb{R}^n)$ for all $1 \le p < n$.*

*iii) If $\int_{\mathbb{R}^n} V(x)\phi(x)\,dx = 0$ and $\int_{\mathbb{R}^n} xV(x)\phi(x)\,dx = 0$ for all zero-energy eigenfunctions $\phi$, then the wave operators extend to bounded operators on $L^p(\mathbb{R}^n)$ for all $1 \le p < \infty$.*

Except when $n = 6$, one can extend the arguments presented here to show that the endpoint case $p = \infty$ holds if one has the additional cancellation $\int_{\mathbb{R}^n} x^2V(x)\phi(x)\,dx = 0$ and slightly more decay on the potential, see Remark 3.5 below. The $L^p$ bounds can be extended to boundedness as an operator on $W^{k,p}(\mathbb{R}^n)$ for the same range of $1 < p < \infty$ with $0 \le k \le 2$ by a standard argument that shows an equivalence between the norms $\|u\|_{W^{k,p}}$ and $\|(-\Delta + c^2)u\|_{L^p}$ for a sufficiently large constant $c$, see [26].

We prove the results for $W = W_-$, the proof for $W_+$ is similar. Following the approach of Yajima in [25], the starting point is the stationary representation of the wave operator

$$\begin{aligned} Wu &= u - \frac{1}{\pi i}\int_0^\infty \lambda R_V^+(\lambda^2) V [R_0^+(\lambda^2) - R_0^-(\lambda^2)]\, u\, d\lambda \\ &= u - \frac{1}{\pi i}\int_0^\infty \lambda \big[R_0^+(\lambda^2) - R_0^+(\lambda^2) V R_V^+(\lambda^2)\big] V [R_0^+(\lambda^2) - R_0^-(\lambda^2)]\, u\, d\lambda \end{aligned} \tag{4}$$

where $R_0^\pm(\lambda^2) := \lim_{\varepsilon\to 0^+}(H_0 - (\lambda \pm i\varepsilon)^2)^{-1}$ and $R_V^+(\lambda^2) := \lim_{\varepsilon\to 0^+}(H - (\lambda + i\varepsilon)^2)^{-1}$ are the free and perturbed resolvents, respectively. These operators are known to be well-defined on polynomially weighted $L^2(\mathbb{R}^n)$ spaces due to the limiting absorption principle, [2]. In dimensions $n > 2$, the free resolvent operators $R_0^\pm(\lambda^2)$ are bounded as $\lambda \to 0$, as are the perturbed resolvents $R_V^\pm(\lambda^2)$ if zero is regular. When zero is not regular, the perturbed resolvent becomes singular as $\lambda \to 0$. This singular behavior shrinks the range of $p$ on which the wave operators are $L^p(\mathbb{R}^n)$ bounded.

The last equality in (4) follows from the standard resolvent identity $R_V^+(\lambda^2) = R_0^+(\lambda^2) - R_0^+(\lambda^2) V R_V^+(\lambda^2)$. One can then split $W$ into high and low energy parts, $W = W\Phi^2(H_0) + W\Psi^2(H_0)$ with $\Phi, \Psi \in C_0^\infty(\mathbb{R})$ smooth cut-off functions that satisfy $\Phi^2(\lambda) + \Psi^2(\lambda) = 1$ with $\Phi(\lambda^2) = 1$ for $|\lambda| \leq \lambda_0/2$ and $\Phi(\lambda^2) = 0$ for $|\lambda| \geq \lambda_0$ for a suitable constant $0 < \lambda_0 \ll 1$. This allows us to write $W = W_< + W_>$, with $W_<$ the 'low energy' portion of the wave operator and $W_>$ the 'high energy' portion. Taking advantage of the intertwining property, one can express $W_> = \Psi(H) W \Psi(H_0)$ and $W_< = \Phi(H) W \Phi(H_0)$.

The weighted Fourier bound on the potential, (3), can be interpreted as requiring a certain amount of smoothness on the potential $V$. In light of the counterexample to dispersive estimates in [14] and the work in [6], it seems possible that one may be able to require less smoothness on the potential, we do not pursue that issue here.

In [25], it was shown that $W_>$ is bounded in $L^p(\mathbb{R}^n)$ for the full range of $1 \leq p \leq \infty$ provided $|V(x)| \lesssim \langle x\rangle^{-n-2-}$ and (3) holds. The high energy portion is unaffected by zero energy eigenvalues.

The fact that $n \geq 5$ allows for greater uniformity in the treatment of $W_<$, as there are no special considerations related to the distinction between resonances and eigenvalues at zero. There are, however, significant differences in the low-energy expansion of the resolvent depending on whether $n$ is even or odd, with the even dimensions presenting more technical challenges due to some logarithmic behavior near zero. The low energy analysis becomes progressively more idiosyncratic for small $n$, requiring additional arguments here for $n = 5, 6, 8, 10$ in particular.

Some results are also known when $n = 4$ in the case where zero is an eigenvalue but not a resonance. The operator $W_<$ is shown in [16] to be bounded on $L^p(\mathbb{R}^4)$ for $\frac{4}{3} < p < 4$, and was recently extended the range to $1 \leq p < 4$ by the authors in [13]. Questions about the $L^p$ boundedness of the wave operators remain open if there is a resonance in four dimensions, or any kind of zero energy obstruction in two dimensions.

The next section sketches an argument that controls the leading order expression for $W_<$ when there is a zero energy eigenvalue. We examine the integral kernel of this operator in order to determine the range of exponents $p$ for which it is bounded on $L^p(\mathbb{R}^n)$. The argument relies on several important integral estimates whose proofs are provided later in Section 4. In Section 3 we show that the argument is direct and flexible enough to be modified to take advantage of the additional cancellation in the event that $P_eV1 = 0$, and more so if $P_eVx = 0$ as well. The summary proof of Theorem 1.1 is given immediately afterward. Section 4 contains a full proof of the key integral estimates. Finally in section 5 we address some additional modifications that are necessary when $n = 6, 8, 10$ in order to control terms of $W_<$ which are not leading order but nevertheless require further scrutiny.

## 2. No cancellation

When there is a zero energy eigenvalue, the perturbed resolvent $R_V^+(\lambda^2)$ in (4) has a pole of order two whose residue is the finite-rank projection $P_e$ onto the eigenspace. The leading term in a low energy expansion for $W_<$ is therefore given by the operator

$$W_{s,2} = \frac{1}{\pi i}\int_0^\infty R_0^+(\lambda^2)VP_eV(R_0^+(\lambda^2) - R_0^-(\lambda^2))\tilde{\Phi}(\lambda)\lambda^{-1}\,d\lambda.$$

In this section we obtain pointwise bounds on the integral kernel of $W_{s,2}$ in order to determine the range of $p$ for which it is $L^p$ bounded.

One can show that the remaining terms in the expansion of $W_<$ are better behaved. Thus the estimates on $W_{s,2}$ dictate the mapping properties of $W_<$ itself. The exact form of the low energy expansion is heavily dependent on whether $n$ is even or odd and is discussed more fully in Section 3 below. The presence or absence of threshold eigenvalues has little effect on properties of the resolvent outside a small neighborhood of $\lambda = 0$, so the estimates for $W_>$ are unchanged.

We first consider this operator under the assumption that there is a zero energy eigenvalue, but no further cancellation. That is, we do not assume that $P_eV1 = 0$ or $P_eVx = 0$.

The kernel of $W_{s,2}$ is a sum of integrals of the form

$$K^{jk}(x,y) = \int_0^\infty \iint_{\mathbb{R}^{2n}} R_0^+(\lambda^2)(x,z)V(z)\phi_j(z)V(w)\phi_k(w) \tag{5}$$
$$(R_0^+ - R_0^-)(\lambda^2)(w,y)\frac{\tilde{\Phi}(\lambda)}{\lambda}\, dwdz\, d\lambda$$

where the functions $\{\phi_j\}_{j=1}^N$ form an orthonormal basis for the zero energy eigenspace, and $\tilde{\Phi}(\lambda) \in C_c^\infty(\mathbb{R})$ is such that $\tilde{\Phi}(\lambda)\Phi(\lambda^2) = \Phi(\lambda^2)$. In [25, 26], Yajima converts the integrals to one-dimensional integrals and proves the desired $L^p$ bounds using the harmonic analysis tools of $A_p$ weights, maximal functions and Hilbert transforms. We approach the same problem by estimating the integrals in $\mathbb{R}^n$ directly. This allows us to recover Yajima's result for $W_{s,2}$ while also obtaining the $p = 1$ endpoint. The intermediate steps can be modified to improve the range of $p$ if there is adequate cancellation, which we show in Section 3.

For the remainder of the paper, we omit the subscripts on the zero-energy eigenfunctions as our calculations will be satisfied for any such $\phi$. Our main estimates are therefore stated for an operator kernel $K(x,y)$ with the understanding that each $K^{jk}(x,y)$ obeys the same bounds. We only utilize the natural decay of $V(z)\phi(z)$ and (later in Section 3) the cancellation hypotheses in Theorem 1.1 which hold for every $\phi$ in the zero energy eigenspace. We first describe the natural decay of zero-energy eigenfunctions.

**Lemma 2.1.** *If $|V(x)| \lesssim \langle x\rangle^{-2-\epsilon}$ for some $\epsilon > 0$, and $\phi$ is a zero-energy eigenfunction, then $|\phi(x)| \lesssim \langle x\rangle^{2-n}$.*

*Proof.* We note from Lemma 5.2 of [11], that any eigenfunction $\phi \in L^\infty(\mathbb{R}^n)$. We then rewrite $(-\Delta+V)\phi = 0$ as $(I+(-\Delta)^{-1}V)\phi = 0$. It is well-known that $(-\Delta)^{-1}$ is an integral operator with integral kernel $c_n|x-y|^{2-n}$. Thus, we have

$$|\phi(x)| = \left|c_n \int_{\mathbb{R}^n} \frac{V(y)\phi(y)}{|x-y|^{n-2}}\, dy\right| \lesssim \|\phi\|_\infty \int_{\mathbb{R}^n} \frac{\langle y\rangle^{-2-\epsilon}}{|x-y|^{n-2}}\, dy \lesssim \langle x\rangle^{-\epsilon}.$$

The last integral bound is easily proven, see for example Lemma 3.8 of [14]. This estimate allows us to bootstrap, increasing the decay of $\phi$ at each step.

$$|\phi(x)| = \left|c_n \int_{\mathbb{R}^n} \frac{V(y)\phi(y)}{|x-y|^{n-2}}\, dy\right| \lesssim \int_{\mathbb{R}^n} \frac{\langle y\rangle^{-2-2\epsilon}}{|x-y|^{n-2}}\, dy \lesssim \langle x\rangle^{-2\epsilon}.$$

After $\frac{n-2}{\epsilon}$ iterations, one has

$$|\phi(x)| = \left|c_n \int_{\mathbb{R}^n} \frac{V(y)\phi(y)}{|x-y|^{n-2}}\, dy\right| \lesssim \int_{\mathbb{R}^n} \frac{\langle y\rangle^{-n-}}{|x-y|^{n-2}}\, dy \lesssim \langle x\rangle^{2-n}.$$

$\square$

The free resolvents $R_0^\pm(\lambda^2)$ appear in mulitple places within the formula for $W_{s,2}$. They are in fact convolution operators whose kernel (for a given $n$) depends on $\lambda$, $|x-y|$, and the choice of sign. Our starting point for handling (5) is to integrate with respect to $\lambda$ and apply the following bound.

**Lemma 2.2.** *Let $R_0^\pm(\lambda^2, A)$ denote the convolution kernel of $R_0^\pm(\lambda^2)$ evaluated at a point with $|x-y| = A$. For each $j \geq 0$,*

$$\int_0^\infty R_0^+(\lambda^2, A)\partial_B^j\big(R_0^+ - R_0^-\big)(\lambda^2, B)\lambda^{-1}\tilde{\Phi}(\lambda)\,d\lambda \lesssim \begin{cases} \frac{1}{A^{n-2}\langle A\rangle^{n-2+j}} & \text{if } A > 2B \\ \frac{1}{A^{n-2}\langle B\rangle^{n-2+j}} & \text{if } B > 2A \\ \frac{1}{A^{n-2}\langle A\rangle\langle A-B\rangle^{n-3+j}} & \text{if } A \approx B \end{cases}. \tag{6}$$

*This can be written more succinctly as*

$$\int_0^\infty R_0^+(\lambda^2, A)\partial_B^j\big(R_0^+ - R_0^-\big)(\lambda^2, B)\lambda^{-1}\tilde{\Phi}(\lambda)\,d\lambda \lesssim \frac{1}{A^{n-2}\langle A+B\rangle\langle A-B\rangle^{n-3+j}}. \tag{7}$$

To handle some lower order terms in the expansion of $W_<$, we make use a related estimate.

**Corollary 2.3.** *Let $R_0^\pm(\lambda^2, A)$ denote the convolution kernel of $R_0^\pm(\lambda^2)$ evaluated at a point with $|x| = A$. For each $j \geq 0$,*

$$\int_0^\infty R_0^+(\lambda^2, A)\partial_B^j\big(R_0^+ - R_0^-\big)(\lambda^2, B)\tilde{\Phi}(\lambda)\,d\lambda \lesssim \begin{cases} \frac{1}{A^{n-2}\langle A\rangle^{n-1+j}} & \text{if } A > 2B \\ \frac{1}{A^{n-2}\langle B\rangle^{n-1+j}} & \text{if } B > 2A \\ \frac{1}{A^{n-2}\langle A\rangle\langle A-B\rangle^{n-2+j}} & \text{if } A \approx B \end{cases}. \tag{8}$$

*This can be written more succinctly as*

$$\int_0^\infty R_0^+(\lambda^2, A)\partial_B^j\big(R_0^+ - R_0^-\big)(\lambda^2, B)\tilde{\Phi}(\lambda)\,d\lambda \lesssim \frac{1}{A^{n-2}\langle A+B\rangle\langle A-B\rangle^{n-2+j}}. \tag{9}$$

**Remark 2.4.** *The $j = 0$ case is what appears in (5). The $j = 1$ and $j = 2$ cases will be used to gain extra decay if $P_eV1 = 0$ and $P_eVx = 0$ respectively.*

Based on Lemma 2.2, we have

$$|K(x,y)| \lesssim \iint_{\mathbb{R}^{2n}} \frac{|V\phi(z)||V\phi(w)|\,dz\,dw}{|x-z|^{n-2}\langle |x-z| + |y-w|\rangle\langle |x-z| - |y-w|\rangle^{n-3}}.$$

If $V\phi(z)$ and $V\phi(w)$ decay rapidly enough, then this integral will be concentrated primarily when $z$ and $w$ are small.

**Lemma 2.5.** *If $|V(z)| \lesssim \langle z\rangle^{-(n-1)-}$, we have the bound*

$$|K(x,y)| \lesssim \iint_{\mathbb{R}^{2n}} \frac{|V\phi(z)||V\phi(w)|\,dz\,dw}{|x-z|^{n-2}\langle |x-z| + |y-w|\rangle\langle |x-z| - |y-w|\rangle^{n-3}}$$

$$\lesssim \frac{1}{\langle x\rangle^{n-2}\langle |x|+|y|\rangle\langle |x|-|y|\rangle^{n-3}}. \tag{10}$$

We delay the proof of Lemmas 2.2 and 2.5, and Corollary 2.3, to Section 4. To show $L^p(\mathbb{R}^n)$ boundedness of certain integral operators, we show that they have an admissible kernel $K(x,y)$, that is

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)|\,dy+\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)|\,dx<\infty. \tag{11}$$

It is well known that an operator with an admissible kernel is bounded on $L^p(\mathbb{R}^n)$ for all $1\leq p\leq\infty$. We now prove

**Proposition 2.6.** *If $|V(z)|\lesssim\langle z\rangle^{-(n-1)-}$, then the operator with kernel $K(x,y)$ is bounded on $L^p(\mathbb{R}^n)$ for $1\leq p<\frac{n}{2}$.*

*Proof.* In the region where $|x|>2|y|$, (10) shows that $|K(x,y)|\lesssim\langle x\rangle^{4-2n}$. We show that this is an admissible kernel. The integral with respect to $x$ is uniformly bounded, as $\langle x\rangle^{4-2n}$ is integrable provided $n>4$. The integral with respect to $y$ is $\langle x\rangle^{4-2n}\int_{|y|<|x|/2}dy\lesssim\langle x\rangle^{4-n}$, which is uniformly bounded so long as $n\geq 4$.

In the region where $|x|\approx|y|$, (10) shows that $|K(x,y)|\lesssim\langle y\rangle^{1-n}\langle |x|-|y|\rangle^{3-n}$. This is also an admissible kernel. By changing to spherical coordinates, we have

$$\int_{|x|\approx|y|}|K(x,y)|\,dx\lesssim\langle y\rangle^{1-n}|y|^{n-1}\int_{|x|/2}^{2|x|}\frac{dr}{\langle |x|-r\rangle^{n-3}}\lesssim 1$$

since $n>4$.

It is only the region where $|y|>2|x|$ that creates restrictions on the $L^p$-boundedness of the operator $K$. Here (10) asserts that $|K(x,y)|\lesssim\langle x\rangle^{2-n}\langle y\rangle^{2-n}$. This is a bounded operator on $L^p(\mathbb{R}^n)$ if one can take the $L^{p'}$ norm in $y$ and then the $L^p$ norm in $x$ with a finite result. In this case

$$\int_{\mathbb{R}^n}\frac{1}{\langle x\rangle^{(n-2)p}}\Big(\int_{|y|>2|x|}\frac{dy}{\langle y\rangle^{(n-2)p'}}\Big)^{p-1}dx\lesssim\int_{\mathbb{R}^n}\frac{1}{\langle x\rangle^{np-4p+n}}\,dx \tag{12}$$

has a convergent inner integral provided $p'>\frac{n}{n-2}$, or in other words $p<n/2$. The second integral converges if $p(n-4)>0$, which is always true for $n>4$. $\square$

## 3. The cases of $P_eV1=0$ and $P_eVx=0$

As noted above, the operator kernel $K(x,y)$ is bounded on the entire range of $L^p(\mathbb{R}^n)$, $1\leq p\leq\infty$, if one considers only the region where $2|x|>|y|$. The restrictions on $p$ occur on account of integrability concerns for large $y$ alone. Specifically, in the region where

$|y| > 2|x|$, (10) provides the bound $|K(x,y)| \lesssim \langle x\rangle^{2-n}\langle y\rangle^{2-n}$, which clearly decays at the rate $\langle y\rangle^{2-n}$ and no faster.

We show that if the nullspace of $H$ satisfies orthogonality conditions $P_eV1 = 0$ and $P_eVx = 0$, then $K(x,y)$ enjoys more rapid decay at infinity and hence the operator is bounded on an expanded range of $L^p$ spaces.

**Proposition 3.1.** *Assume $P_eV1 = 0$ and $|V(z)| \lesssim \langle z\rangle^{-n-1-}$. The operator with kernel $K(x,y)$ is bounded on $L^p(\mathbb{R}^n)$ for $1 \le p < n$ whenever $n > 4$.*

The original estimate (10) is already sufficient to prove this in the regions where $2|x| > |y|$, and also when $|x|, |y| \le 10$. In fact, it is even possible to extract the desired decay in $\langle y\rangle$ from much of the proof of Lemma 4.3 when $y$ is large. More specifically,

**Lemma 3.2.** *Let $k \ge 0$. If $N \ge 2n-3+k$ and $R \ge 0$ is fixed, then*

$$\int_{|w|>\frac{|y|}{2}} \frac{\langle w\rangle^{-N}}{\langle R+|y-w|\rangle\langle R-|y-w|\rangle^{n-3}}\,dw \lesssim \frac{1}{\langle R+|y|\rangle\langle R-|y|\rangle^{n-3}\langle y\rangle^k} \tag{13}$$

*Proof.* The estimates for (31) and (33) can be reproduced verbatim, using $\alpha = 0$ and $\beta = n-3$, the only change being that $N \ge 2n-3+k \ge n+\beta+k$ instead of $N \ge n+\beta$. On the annulus where $|y-w| \approx |y|$, excluding the ball $|w| < \frac{|y|}{2}$ yields the bound

$$\begin{aligned}\int_{\substack{|y-w|\approx|y|\\ |w|>\frac{|y|}{2}}} \frac{\langle w\rangle^{-N}}{\langle R+|y-w|\rangle\langle R-|y-w|\rangle^{n-3}}\,dw &\lesssim \langle y\rangle^{-N}\int_{\frac{|y|}{2}}^{2|y|} \frac{r^{n-1}}{\langle R+r\rangle\langle R-r\rangle^{n-3}}\,dr\\ &\lesssim \frac{1}{\langle y\rangle^{N-n}\langle R+|y|\rangle}\max_{\frac{|y|}{2}<r<2|y|}\langle r-R\rangle^{3-n}.\end{aligned}$$

If $R < \frac{|y|}{4}$ or $R > 4|y|$, then $\langle r-R\rangle \approx \langle R-|y|\rangle$ over the interval of integration, and (13) is satisfied so long as $N \ge n+k$.

When $R \approx |y|$, the maximum value of $\langle r-R\rangle^{3-n}$ might be 1. In that case $\langle R-|y|\rangle \lesssim \langle y\rangle$, so the integral is bounded by $\langle y\rangle^{n-1-N} \lesssim \langle y\rangle^{2n-3-N}\langle R+|y|\rangle^{-1}\langle R-|y|\rangle^{3-n}$. Then (13) is satisfied so long as $N \ge 2n-3+k$. $\square$

*Proof of Proposition 3.1.* By the discussion preceding Lemma 3.2, we may assume that $|y| > 2|x|$ and $|y| > 10$, and it suffices to demonstrate the bound

$$|K(x,y)| \lesssim \frac{1}{\langle x\rangle^{n-2}\langle y\rangle^{n-1}} \tag{14}$$

in this region.

The cancellation condition $P_eV1 = 0$ allows us to rewrite the $K(x,y)$ integral in the following manner.

$$K(x,y) = \iint_{\mathbb{R}^{2n}} \int_0^\infty V\phi(z)V\phi(w)R_0^+(\lambda^2,|x-z|) \\ \big((R_0^+ - R_0^-)(\lambda^2,|y-w|) - (R_0^+ - R_0^-)(\lambda^2,|y|)\big)\frac{\tilde{\Phi}(\lambda)}{\lambda}\,d\lambda\,dz\,dw. \tag{15}$$

Subtracting the function $(R_0^+ - R_0^-)(\lambda^2,|y|)$, which is independent of $w$, from the integrand does not affect the final value because our standing assumption $P_eV1 = 0$ implies that $\int_{\mathbb{R}^n} V\phi(w)\,dw = 0$ for all eigenfunctions $\phi$.

For any function $F(\lambda,|y|)$ one can express

$$F(\lambda,|y-w|) - F(\lambda,|y|) = \int_0^1 \partial_r F(\lambda,|y-sw|)\frac{(-w)\cdot(y-sw)}{|y-sw|}\,ds. \tag{16}$$

Here we are interested in $F(\lambda,|y|) = (R_0^+ - R_0^-)(\lambda^2,|y|) = (\frac{\lambda}{|y|})^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(\lambda|y|)$, whose radial derivatives are considered in the statement and proof of Lemma 2.2. Which side of the identity (16) we use is decided based on the size of $|w|$ compared to $\frac{1}{2}|y|$. Accordingly, we divide the $w$ integal into two regions. On the region where $|w| < \frac{1}{2}|y|$, the contribution to $K(x,y)$ has the expression

$$\int_{|w|<\frac{|y|}{2}} \int_{\mathbb{R}^n} \int_0^\infty \int_0^1 V\phi(z)V\phi(w)R_0^+(\lambda^2,|x-z|)\partial_r\big((R_0^+ - R_0^-)(\lambda^2,|y-sw|)\big) \\ \frac{(-w)\cdot(y-sw)}{|y-sw|}\frac{\tilde{\Phi}(\lambda)}{\lambda}\,ds\,d\lambda\,dz\,dw, \tag{17}$$

where $\partial_r$ indicates the partial derivative with respect to the radial variable of $R_0^\pm(\lambda^2,r)$. Apply Fubini's Theorem and then Lemma 2.2 with $j=1$ to obtain the upper bound

$$\int_0^1 \int_{|w|<\frac{|y|}{2}} \int_{\mathbb{R}^n} \frac{|V\phi(z)|\,|wV\phi(w)|}{|x-z|^{n-2}\langle |x-z|+|y-sw|\rangle\langle |x-z|-|y-sw|\rangle^{n-2}}\,dz\,dw\,ds.$$

By Lemma 2.1 and our assumption that $|V(z)| \lesssim \langle z\rangle^{-(n+1)-}$, we can control the decay of the numerator with $|V\phi(z)| \lesssim \langle z\rangle^{-(2n-1)-}$ and $|wV\phi(w)| \lesssim \langle w\rangle^{-(2n-2)-}$. These are sufficient to apply Lemma 4.3 in the $z$ variable, then Lemma 4.4 in the $w$ variable to obtain

$$|K(x,y)| \lesssim \int_0^1 \int_{|w|<\frac{|y|}{2}} \frac{|V\phi(w)|}{\langle x\rangle^{n-2}\langle |x|+|y-sw|\rangle\langle |x|-|y-sw|\rangle^{n-2}}\,dw\,ds \\ \lesssim \int_0^1 \frac{1}{\langle x\rangle^{n-2}\langle |x|+|y|\rangle\langle |x|-|y|\rangle^{n-2}}\,ds \lesssim \frac{1}{\langle x\rangle^{n-2}\langle y\rangle^{n-1}}$$

when $|y| > 2|x|$, as desired.

For the portion of (15) where $|w| > \frac{1}{2}|y|$, we treat the two terms in the difference directly instead of rewriting as an integral using (16). For the term with $(R_0^+ - R_0^-)(\lambda^2,|y-w|)$,

direct applications of Lemmas 2.2 and 3.2 with $j = 0$ and $k = 1$ respectively, followed by Lemma 4.3 with respect to the $z$ variable, shows that

$$\int_{\mathbb{R}^n}\int_{|w|>\frac{|y|}{2}}\int_0^\infty R_0^+(\lambda^2,|x-z|)V\phi(z)V\phi(w)(R_0^+-R_0^-)(\lambda^2,|y-w|)\frac{\tilde{\Phi}(\lambda)}{\lambda}\,d\lambda\,dw\,dz$$
$$\lesssim \int_{\mathbb{R}^n}\int_{|w|>\frac{|y|}{2}} \frac{|V\phi(z)|\,|V\phi(w)|}{|x-z|^{n-2}\langle|x-z|+|y-w|\rangle\langle|x-z|-|y-w|\rangle^{n-3}}\,dw\,dz$$
$$\lesssim \frac{1}{\langle x\rangle^{n-2}\langle|x|+|y|\rangle\langle|x|-|y|\rangle^{n-3}\langle y\rangle} \lesssim \frac{1}{\langle x\rangle^{n-2}\langle y\rangle^{n-1}}$$

within the region where $|y| > 2|x|$.

The estimate for the term with $(R_0^+ - R_0^-)(\lambda^2, |y|)$ is more straightforward.

$$\int_{\mathbb{R}^n}\int_{|w|>\frac{|y|}{2}}\int_0^\infty R_0^+(\lambda^2,|x-z|)V\phi(z)V\phi(w)(R_0^+-R_0^-)(\lambda^2,|y|)\frac{\tilde{\Phi}(\lambda)}{\lambda}\,d\lambda\,dw\,dz$$
$$\lesssim \int_{\mathbb{R}^n}\int_{|w|>\frac{|y|}{2}} \frac{|V\phi(z)|\,|V\phi(w)|}{|x-z|^{n-2}\langle|x-z|+|y|\rangle\langle|x-z|-|y|\rangle^{n-3}}\,dw\,dz$$
$$\lesssim \frac{1}{\langle x\rangle^{n-2}\langle|x|+|y|\rangle\langle|x|-|y|\rangle^{n-3}\langle y\rangle} \lesssim \frac{1}{\langle x\rangle^{n-2}\langle y\rangle^{n-1}}.$$

We have used Lemma 4.3 for the integration in $z$, and the basic estimate $\int_{|w|>|y|/2}\langle w\rangle^{-N}\,dw \lesssim \langle y\rangle^{n-N}$ in lieu of Lemma 3.2. We can now run through the argument as in (12) to see that the extra decay in $y$ allows the resulting integral to converge provided $(n-1)p' > n$, which requires $p < n$. □

Finally, we show that with further cancellation, one can extend to nearly the full range of $p$. That is,

**Proposition 3.3.** *Assume $P_eV1 = 0$, $P_eVx = 0$ and $|V(z)| \lesssim \langle z\rangle^{-n-3-}$, then the operator with kernel $K(x, y)$ is bounded on $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$ whenever $n > 4$.*

*Proof.* We note that when $P_eV1, P_eVx = 0$, we have the equality

$$(18)\quad \int_{\mathbb{R}^n}\int_0^\infty R_0^+(\lambda^2)(x,z)V(z)\phi(z)V(w)\phi(w)(R_0^+-R_0^-)(\lambda^2)(w,y)\frac{\tilde{\Phi}(\lambda)}{\lambda}\,d\lambda\,dw$$
$$= \int_{\mathbb{R}^n}\int_0^\infty R_0^+(\lambda^2)(x,z)V\phi(z)V\phi(w)\Big[(R_0^+-R_0^-)(\lambda^2)(w,y)-F(\lambda,y)-G(\lambda,y)\frac{w\cdot y}{|y|}\Big]\frac{\tilde{\Phi}(\lambda)}{\lambda}\,d\lambda\,dw,$$

for any functions $F(\lambda, y)$ and $G(\lambda, y)$. In place of (16), we utilize the extra level of cancellation to write

$$(19)\quad K(\lambda,|y-w|) - K(\lambda,|y|) + \partial_r K(\lambda,|y|)\frac{w\cdot y}{|y|}$$

$$= \int_0^1 (1-s)\Big[\partial_r^2 K(\lambda, |y-sw|)\frac{(w\cdot(y-sw))^2}{|y-sw|^2} \\ + \partial_r K(\lambda,|y-sw|)\Big(\frac{|w|^2}{|y-sw|} - \frac{(w\cdot(y-sw))^2}{|y-sw|^3}\Big)\Big]\,ds.$$

The formula above suggests that we choose $F(\lambda, y) = (R_0^+ - R_0^-)(\lambda^2, |y|)$ and $G(\lambda, y) = \partial_r(R_0^+ - R_0^-)(\lambda^2, |y|)$ in (18) respectively.

As in the proof of Proposition 3.1, whenever $|w| > |y|/2$, we can use the decay of $V\phi(w)$ to limit its contribution to $K(x, y)$ to a term of size $\langle x\rangle^{2-n}\langle y\rangle^{-n}$ when $|y| > 2|x|$.

If, on the other hand, $|w| < |y|/2$, there are new terms to bound of the form

$$\int_{|w|<\frac{|y|}{2}} \int_{\mathbb{R}^n} \int_0^\infty \int_0^1 V\phi(z)V\phi(w)R_0^+(\lambda^2, |x-z|)\partial_r^j\big((R_0^+ - R_0^-)(\lambda^2, |y-sw|)\big) \\ (1-s)\Gamma_j(s, w, y)\frac{\tilde{\Phi}(\lambda)}{\lambda}\, ds\, d\lambda\, dz\, dw$$

with $j = 1, 2$ and $\Gamma_j(s, w, y)$ denoting

$$\Gamma_1 = \Big(\frac{|w|^2}{|y-sw|} - \frac{(w\cdot(y-sw))^2}{|y-sw|^3}\Big), \qquad \text{and} \qquad \Gamma_2 = \frac{(w\cdot(y-sw))^2}{|y-sw|^2}.$$

Within the range $|w| < |y|/2$ and $0 \le s \le 1$, these factors observe the bounds $|\Gamma_1(s, w, y)| \lesssim |y|^{-1}|w|^2$ and $|\Gamma_2(s, w, y)| \le |w|^2$. The calculation continues in the same manner as in Proposition 3.1, first using Lemma 2.2 with $j = 1, 2$, then Lemma 4.3 in the $z$ integral and Lemma 4.4 (with $\alpha = 2 - j$) in the $w$ integral. For both values of $j$ we arrive at the bound $\langle x\rangle^{2-n}\langle y\rangle^{-n}$ when $|y| > 2|x|$.

Put together with the previous claim, this implies that in fact

$$|K(x, y)| \lesssim \frac{1}{\langle x\rangle^{n-2}\langle y\rangle^n} \text{ when } |y| > 2|x|. \tag{20}$$

The estimates from (10) sill hold when $|x| \approx |y|$ and $|x| > 2|y|$. In the region where $|y| > 2|x|$ we can imitate the calculation in (12) and find convergent integrals so long as $p' > 1$. The operator with kernel $K(x, y)$ is therefore bounded on $L^p(\mathbb{R}^n)$ for all $p \in [1, \infty)$.

The extra growth of $|w|^2$ in the size of $\Gamma_j(s, w, y)$ dictates the amount of decay we need on the potential. We must use Lemma 4.4 with $\beta = n-3+j$, which requires $|w|^2|V\phi(w)| \lesssim \langle w\rangle^{1-2n}$, from which Lemma 2.1 shows that $|V(w)| \lesssim \langle w\rangle^{-n-3-}$ is needed. □

**Remark 3.4.** *It appears to be possible to make an analogous cancellation argument to improve decay with respect to $x$ in the region $|x| > 2|y|$. Given the existing strength of* (10)

*here, the benefits of doing so are unclear. When $|x| \approx |y|$, cancellation only leads to improvement in the exponent of $\langle |x| - |y| \rangle$, which does not affect the integrability properties of $K(x, y)$ in a meaningful way.*

We can now prove the main Theorem.

*Proof of Theorem 1.1.* We first prove the desired results for $n > 3$ odd. In this case, one has the expansion $W_< = \Phi(H)(1 - (W_{r,0} + W_r + W_{s,1} + W_{s,2}))\Phi(H_0)$, where

$$\begin{aligned} W_{r,0} &= \frac{1}{\pi i} \int_0^\infty R_0^+(\lambda^2) V (R_0^+(\lambda^2) - R_0^-(\lambda^2)) \lambda \, d\lambda \\ W_r &= \frac{1}{\pi i} \int_0^\infty R_0^+(\lambda^2) V A_0(\lambda) (R_0^+(\lambda^2) - R_0^-(\lambda^2)) \tilde{\Phi}(\lambda) \lambda \, d\lambda \\ W_{s,1} &= \frac{1}{\pi i} \int_0^\infty R_0^+(\lambda^2) V A_{-1} (R_0^+(\lambda^2) - R_0^-(\lambda^2)) \tilde{\Phi}(\lambda) \, d\lambda \\ W_{s,2} &= \frac{1}{\pi i} \int_0^\infty R_0^+(\lambda^2) V P_e V (R_0^+(\lambda^2) - R_0^-(\lambda^2)) \tilde{\Phi}(\lambda) \lambda^{-1} \, d\lambda \end{aligned} \tag{21}$$

Here the subscripts $r$ denotes regular and $s$ denotes singular terms. In [25], Yajima shows that the first two 'regular' terms are bounded on $L^p$ for $1 \leq p \leq \infty$. In particular, it is shown that $\|W_{r,0} u\|_p \lesssim \|\mathcal{F}(\langle \cdot \rangle^\sigma V)\|_{L^{n_*}} \|u\|_p$ for $\sigma > \frac{1}{n_*}$.

Under the assumptions on the decay of $V$ and (3), it was shown in [25] that $W_>$, $\Phi(H)(1 - (W_{r,0} + W_r))\Phi(H_0)$ are all bounded on $L^p$ for $1 \leq p \leq \infty$. Further, $W_{s,1} = 0$ if $n > 5$. We note that, if $n = 5$ and $P_e V 1 = 0$ that $W_{s,1} = 0$, Section 3.2.2 in [25]. Thus, we need only bound $W_{s,2}$ for all $n > 4$.

The kernels of $\Phi(H)$ and $\Phi(H_0)$ are bounded by $C_N \langle x - y \rangle^{-N}$ for each $N = 1, 2, \dots$, see Lemma 2.2 of [23]. Following (21), (5), $W_{s,2}$ is bounded on $L^p$ exactly when the operators $K^{jk}$ are. Proposition 2.6 proves the first claim for all $n > 5$, Proposition 3.1 proves the second, while Proposition 3.3 proves the third.

We need to make a few adjustments to our approach when $n = 5$ if there is no cancellation. The operator $W_{s,1}$ may be rewritten as

$$\frac{1}{\pi i} \int_0^\infty R_0^+(\lambda^2) [V(P_0 V) \otimes V(P_0 V)] (R_0^+(\lambda^2) - R_0^-(\lambda^2)) \tilde{\Phi}(\lambda) \, d\lambda,$$

where $P_0 V = \sum \phi_j \langle V, \phi_j \rangle$ is a function with the same decay properties as an eigenfunction in Lemma 2.1. The $L^p$ boundedness follows by using Corollary 2.3 and modifying Proposition 2.6 accordingly.

The proof for $n$ even is quite similar. We note that [26, Section 2.2] allows us to express $W_< = \Phi(H)(W_r + W_{\log} + W_{s,2})\Phi(H_0)$. When $n > 10$, it is shown there that $W_{\log}$ vanishes

and $W_r$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$. These results, plus our Proposition 2.6, establish the first claim of Theorem 1.1. Noting equation (2.19) in [26], if $n > 6$ and $P_eV1 = 0$, the operator $W_{\log}$ vanishes, so Proposition 3.1 proves the second claim, while Proposition 3.3 proves the third.

When $n = 8, 10$, one also needs to control the contribution of $W_{\log}$ in order to prove the first claim of Theorem 1.1. When $n = 6$ it is not clear that $W_{\log}$ vanishes under any of the given cancellation conditions. We show that it is bounded on $L^p(\mathbb{R}^6)$ for all $1 \leq p < \infty$, which is sufficient to complete the proof for all three claims in Theorem 1.1. The additional arguments require some modification of the main techniques used and are presented separately in Section 5 below.

□

**Remark 3.5.** *We note that the endpoint $p = \infty$ is not covered in our analysis when $P_eV1, P_eVx = 0$. If, we add in the additional assumption $P_eVx^2 = 0$ and $|V(z)| \lesssim \langle z\rangle^{-n-5-}$, we can use the techniques above to show that the wave operators are bounded on $L^p(\mathbb{R}^n)$ for the full range of $1 \leq p \leq \infty$. Here the assumption $P_eVx^2 = 0$ means that $\int_{\mathbb{R}^n} P_2(x)V(x)\phi(x)\,dx = 0$ for any quadratic monomial $P_2$. We leave the details to the reader.*

## 4. Integral Estimates

4.1. **Proof of Lemma 2.2.** We first recall the inequality claimed in Lemma 2.2, namely

$$\int_0^\infty R_0^+(\lambda^2, A)\partial_B^j\big(R_0^+ - R_0^-\big)(\lambda^2, B)\lambda^{-1}\tilde{\Phi}(\lambda)\,d\lambda \lesssim \begin{cases} \frac{1}{A^{n-2}\langle A\rangle^{n-2+j}} & \text{if } A > 2B \\ \frac{1}{A^{n-2}\langle B\rangle^{n-2+j}} & \text{if } B > 2A \\ \frac{1}{A^{n-2}\langle A\rangle\langle A-B\rangle^{n-3+j}} & \text{if } A \approx B \end{cases} \cdot$$

The proof is a lengthy exercise in integration by parts. The following elementary bound will be invoked repeatedly.

**Lemma 4.1.** *Suppose there exists $\beta > -1$ and $M > \beta + 1$ such that $|F^{(k)}(\lambda)| \lesssim \lambda^{\beta-k}$ for all $0 \leq k \leq M$. Then given a smooth cutoff function $\tilde{\Phi}$,*

$$\Big|\int_0^\infty e^{i\rho\lambda}F(\lambda)\tilde{\Phi}(\lambda)\,d\lambda\Big| \lesssim \langle\rho\rangle^{-\beta-1}. \tag{22}$$

*If $F$ is further assumed to be smooth and supported in the annulus $L \lesssim \lambda \lesssim 1$ for some $L > \rho^{-1} > 0$, then*

$$\Big|\int_0^\infty e^{i\rho\lambda}F(\lambda)\tilde{\Phi}(\lambda)\,d\lambda\Big| \lesssim \langle\rho\rangle^{-M}L^{\beta+1-M}. \tag{23}$$

*Proof.* It is immediately true that (22) is uniformly bounded. Assume $\rho \gtrsim 1$. Then

$$\Big| \int_0^{\rho^{-1}} e^{i\rho\lambda} F(\lambda)\tilde{\Phi}(\lambda)\, d\lambda \Big| \lesssim \int_0^{\rho^{-1}} \lambda^\beta\, d\lambda \lesssim \rho^{-\beta-1}.$$

By repeated integration by parts,

$$\Big| \int_{\rho^{-1}}^{\infty} e^{i\rho\lambda} F(\lambda)\tilde{\Phi}(\lambda)\, d\lambda \Big| \lesssim \sum_{k=1}^{M} \rho^{-k} |F^{(k-1)}(\rho^{-1})| + \rho^{-M} \int_{\rho^{-1}}^{\infty} \Big| \big(\tfrac{d}{d\lambda}\big)^M (F\tilde{\Phi})(\lambda) \Big|\, d\lambda \lesssim \rho^{-\beta-1}.$$

We also use the fact that $|\tilde{\Phi}^{(k)}(\lambda)| \lesssim \lambda^{-k}$ in the last inequality.

In the second case, the integral over $0 \le \lambda \le \rho^{-1}$ is empty, and no boundary terms are created by the integration by parts. Then

$$\Big| \int_0^{\infty} e^{i\rho\lambda} F(\lambda)\tilde{\Phi}(\lambda)\, d\lambda \Big| \lesssim \rho^{-M} \int_L^{\infty} \big(\tfrac{d}{d\lambda}\big)^M (F\tilde{\Phi})(\lambda)\, d\lambda \lesssim \rho^{-M} L^{\beta+1-M}.$$

$\square$

*Proof of Lemma 2.2.* For complex $z \in \mathbb{C} \setminus [0,\infty)$, the free resolvent $(H_0 - z)^{-1}$ is a convolution operator whose kernel may be expressed in terms of Hankel functions.

$$R_0(z)(x,y) = \frac{i}{4} \left( \frac{z^{1/2}}{2\pi|x-y|} \right)^{\frac{n}{2}-1} H^{(1)}_{\frac{n}{2}-1}(z^{1/2}|x-y|). \tag{24}$$

Here $H^{(1)}_{\frac{n}{2}-1}(\cdot)$ is the Hankel function of the first kind and $z^{1/2}$ is defined to take values in the upper halfplane. The Hankel functions of half-integer order, which arise when $n$ is odd, can be expressed more simply as the product of an exponential function and a polynomial.

Formula (24) yields a set of asymptotic descriptions of $R_0^{\pm}(\lambda^2, A)$, based on the properties of Hankel and Bessel functions in [1]. We will use the fact that $R_0^{\pm}(\lambda^2, A) \approx A^{2-n}$ when $\lambda A \lesssim 1$, and $R_0^{\pm}(\lambda^2, A) = e^{\pm i\lambda A} A^{2-n} \Psi_{\frac{n-3}{2}}(\lambda A)$ when $\lambda A \gtrsim 1$. In odd dimensions, the function $\Psi_{\frac{n-3}{2}}$ is actually a polynomial of order $\frac{n-3}{2}$. In even dimensions it is a function that asymptotically behaves like $(\,\cdot\,)^{(n-3)/2}$ and whose $k^{th}$ derivative behaves like $(\,\cdot\,)^{(n-3-2k)/2}$.

For the difference of resolvents, we have the low-energy description $R_0^+(\lambda^2, B) - R_0^-(\lambda^2, B) = (\frac{\lambda}{B})^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(\lambda B) \approx \lambda^{n-2}$. This is a real-analytic function of $\lambda$ and $B$ so it has well behaved derivatives of all orders. We concisely describe the resolvent kernels for $n \ge 3$ and their differences as follows:

$$R_0^{\pm}(\lambda^2, A) = \frac{1}{A^{n-2}} \Omega(\lambda A) + \frac{e^{\pm i\lambda A}}{A^{n-2}} \Psi_{\frac{n-3}{2}}(\lambda A), \tag{25}$$

$$R_0^+(\lambda^2, B) - R_0^-(\lambda^2, B) = \lambda^{n-2}\Omega(\lambda B) + \frac{e^{i\lambda B}}{B^{n-2}} \Psi_{\frac{n-3}{2}}(\lambda B) + \frac{e^{-i\lambda B}}{B^{n-2}} \Psi_{\frac{n-3}{2}}(\lambda B), \tag{26}$$

$$\partial_B^j\big(R_0^+(\lambda^2,B)-R_0^-(\lambda^2,B)\big)=\lambda^{n-2+j}\Omega(\lambda B)+\frac{e^{i\lambda B}}{B^{n-2}}\lambda^j\Psi_{\frac{n-3}{2}}(\lambda B)+\frac{e^{-i\lambda B}}{B^{n-2}}\lambda^j\Psi_{\frac{n-3}{2}}(\lambda B). \tag{27}$$

where $\Omega$ is a bounded compactly supported function that is smooth everywhere except possibly at zero, and each $\Psi_{\frac{n-3}{2}}$ is a smooth function supported outside the unit interval with the asymptotic behavior specified above. In odd dimensions, the polynomial $\Psi_{\frac{n-3}{2}}(\lambda B)$ in (27) may include negative powers of order up to $(\lambda B)^{-j}$. Since the support of $\Psi_{\frac{n-3}{2}}$ is bounded away from zero, the presence of negative powers is of little consequence.

With some abuse of notation, we use $\Omega$ and $\Psi_{\frac{n-3}{2}}$ to describe several distinct functions. It should be understood that all we need from these functions are the properties that this notation yields. As a relevant example, we say that $\frac{d}{d\lambda}\Psi_{\frac{n-3}{2}}(\lambda A) = A\Psi_{\frac{n-5}{2}}(\lambda A) = \lambda^{-1}\Psi_{\frac{n-3}{2}}(\lambda A)$ and $\frac{d}{dA}\Psi_{\frac{n-3}{2}}(\lambda A) = \lambda\Psi_{\frac{n-5}{2}}(\lambda A) = A^{-1}\Psi_{\frac{n-3}{2}}(\lambda A)$. The new functions $\Psi_{\frac{n-3}{2}}$ retain the general properties of the original, but with different constants and/or coefficients.

First case: $A > 2B$. We split the integral into four pieces:

$$\begin{aligned}\int_0^\infty \bigg(\frac{\lambda^{n-3+j}}{A^{n-2}}\Omega(\lambda A)\Omega(\lambda B) &+ \frac{e^{i\lambda A}}{A^{n-2}}\lambda^{n-3+j}\Psi_{\frac{n-3}{2}}(\lambda A)\Omega(\lambda B)\\ &+\frac{e^{i\lambda(A-B)}}{A^{n-2}B^{n-2}}\lambda^{j-1}\Psi_{\frac{n-3}{2}}(\lambda A)\Psi_{\frac{n-3}{2}}(\lambda B)\\ &+\frac{e^{i\lambda(A+B)}}{A^{n-2}B^{n-2}}\lambda^{j-1}\Psi_{\frac{n-3}{2}}(\lambda A)\Psi_{\frac{n-3}{2}}(\lambda B)\bigg)\tilde{\Phi}(\lambda)\,d\lambda.\end{aligned} \tag{28}$$

The first term is supported where $\lambda \lesssim \langle A\rangle^{-1}$ and is bounded pointwise by $A^{2-n}\lambda^{n-3+j}$, so its integral is clearly dominated by $A^{2-n}\langle A\rangle^{2-n-j}$. All other terms are empty unless $A \gtrsim 1$.

The second term of (28) is supported in an interval with $\frac{1}{A} \lesssim \lambda \lesssim \frac{1}{B}$, and goes to zero smoothly at both endpoints. We can write

$$\lambda^{n-3+j}\Psi_{\frac{n-3}{2}}(\lambda A)\Omega(\lambda B) = A^{\frac{n-3}{2}}F(\lambda).$$

Observe that $F(\lambda)$ satisfies the conditions of Lemma 4.1 with $\beta = \frac{3n-9}{2}+j$, and constants uniform in $A$ and $B$. If any derivatives fall on $\Omega(\lambda B)$, the resulting factor of $B$ is dominated by $\lambda^{-1}$ inside the support of $F(\lambda)$. Thus

$$\int_0^\infty \frac{e^{i\lambda A}}{A^{n-2}}\lambda^{n-3+j}\Psi_{\frac{n-3}{2}}(\lambda A)\Omega(\lambda B)\tilde{\Phi}(\lambda)\,d\lambda \lesssim \frac{1}{A^{\frac{n-1}{2}}\langle A\rangle^{\frac{3n-7}{2}+j}} \lesssim \frac{1}{A^{n-2}\langle A\rangle^{n-2+j}}.$$

The third and fourth terms of (28) are quite similar, so we will consider only the third term here. This is supported in an interval with $\frac{1}{B} \lesssim \lambda \lesssim 1$ and goes smoothly to zero at

both endpoints. We can write

$$\lambda^{j-1}\Psi_{\frac{n-3}{2}}(\lambda A)\Psi_{\frac{n-3}{2}}(\lambda B) = A^{\frac{n-3}{2}}B^{\frac{n-3}{2}}F(\lambda),$$

where $F(\lambda)$ satisfies the stronger conditions of Lemma 4.1 with $\beta = n-4+j$ and $L = B^{-1}$. Thus

$$\Big|\int_0^\infty \frac{e^{i\lambda(A-B)}}{A^{n-2}B^{n-2}}\lambda^{j-1}\Psi_{\frac{n-3}{2}}(\lambda A)\Psi_{\frac{n-3}{2}}(\lambda B)\tilde{\Phi}(\lambda)\,d\lambda\Big| \lesssim \frac{1}{A^{\frac{n-1}{2}}B^{\frac{3n-7}{2}+j-M}\langle A-B\rangle^M} \lesssim \frac{1}{A^{n-2}\langle A\rangle^{n-2+j}},$$

using the assumptions $A > 2B$ (so that $\langle A-B\rangle \approx \langle A\rangle$), $A \gtrsim 1$, and choosing $M \geq \frac{3n-7}{2}+j$. The fourth term is less delicate, as we have identical bounds with $\langle A+B\rangle$ in place of $\langle A-B\rangle$.

Second case: $B > 2A$. This time the integral splits into five terms:

$$\begin{aligned}(29)\quad \int_0^\infty \Big(&\frac{\lambda^{n-3+j}}{A^{n-2}}\Omega(\lambda A)\Omega(\lambda B) + \frac{e^{i\lambda B}}{(AB)^{n-2}}\lambda^{j-1}\Psi_{\frac{n-3}{2}}(\lambda B)\Omega(\lambda A)\\ &+ \frac{e^{-i\lambda B}}{(AB)^{n-2}}\lambda^{j-1}\Psi_{\frac{n-3}{2}}(\lambda B)\Omega(\lambda A) + \frac{e^{i\lambda(A-B)}}{(AB)^{n-2}}\lambda^{j-1}\Psi_{\frac{n-3}{2}}(\lambda A)\Psi_{\frac{n-3}{2}}(\lambda B)\\ &+ \frac{e^{i\lambda(A+B)}}{(AB)^{n-2}}\lambda^{j-1}\Psi_{\frac{n-3}{2}}(\lambda A)\Psi_{\frac{n-3}{2}}(\lambda B)\Big)\tilde{\Phi}(\lambda)\,d\lambda.\end{aligned}$$

The first term is supported where $\lambda \lesssim \langle B\rangle^{-1}$ and is bounded pointwise by $A^{2-n}\lambda^{n-3+j}$, so its integral is dominated by $A^{2-n}\langle B\rangle^{2-n-j}$ as desired. All other terms are empty unless $B \gtrsim 1$.

The second and third terms are similar, so we will only consider the second term here. We can write

$$\lambda^{j-1}\Psi_{\frac{n-3}{2}}(\lambda B)\Omega(\lambda A) = B^{\frac{n-3}{2}}F(\lambda),$$

where $F(\lambda)$ satisfies the conditions of Lemma 4.1 with $\beta = \frac{n-5}{2}+j$. Note that $A \lesssim \lambda^{-1}$ inside the support of $\Omega(\lambda A)$. As a consequence,

$$\Big|\int_0^\infty \frac{e^{i\lambda B}}{(AB)^{n-2}}\lambda^{j-1}\Psi_{\frac{n-3}{2}}(\lambda B)\Omega(\lambda A)\tilde{\Phi}(\lambda)\,d\lambda\Big| \lesssim \frac{1}{A^{n-2}B^{\frac{n-1}{2}}\langle B\rangle^{\frac{n-3}{2}+j}} \lesssim \frac{1}{A^{n-2}\langle B\rangle^{n-2+j}}$$

since $B \gtrsim 1$.

The fourth and fifth terms are also very similar, and we consider only the fourth term here. The integrand is supported in an interval with $\frac{1}{A} \lesssim \lambda \lesssim 1$ and goes smoothly to zero at both endpoints. We can write

$$\lambda^{j-1}\Psi_{\frac{n-3}{2}}(\lambda A)\Psi_{\frac{n-3}{2}}(\lambda B) = (AB)^{\frac{n-3}{2}}F(\lambda),$$

where $F(\lambda)$ satisfies the stronger conditions of Lemma 4.1 with $\beta = n-4+j$ and $L = \frac{1}{A}$. It follows from (23) that

$$\Big| \int_0^\infty \frac{e^{i\lambda(A-B)}}{(AB)^{n-2}} \lambda^{j-1} \Psi_{\frac{n-3}{2}}(\lambda A)\Psi_{\frac{n-3}{2}}(\lambda B)\tilde{\Phi}(\lambda)\Big| \lesssim \frac{1}{A^{\frac{3n-7}{2}+j-M} B^{\frac{n-1}{2}} \langle B\rangle^M} \lesssim \frac{1}{A^{n-2}\langle B\rangle^{n-2+j}}$$

using the assumption that $B > 2A$ and choosing $M \geq n-3+j > \frac{n-3}{2} + j$.

Third case: $A \approx B$. If one is overly cautious, there might be as many as six terms to consider in the integral.

$$\begin{aligned} \int_0^\infty \Big( &\frac{\lambda^{n-3+j}}{A^{n-2}} \Omega(\lambda A)\Omega(\lambda B) + \frac{e^{i\lambda A}}{A^{n-2}} \lambda^{n-3+j} \Psi_{\frac{n-3}{2}}(\lambda A)\Omega(\lambda B) \\ &+ \frac{e^{i\lambda B}}{(AB)^{n-2}} \lambda^{j-1} \Psi_{\frac{n-3}{2}}(\lambda B)\Omega(\lambda A) + \frac{e^{-i\lambda B}}{(AB)^{n-2}} \lambda^{j-1} \Psi_{\frac{n-3}{2}}(\lambda B)\Omega(\lambda A) \\ &+ \frac{e^{i\lambda(A-B)}}{A^{n-2}B^{n-2}} \lambda^{j-1} \Psi_{\frac{n-3}{2}}(\lambda A)\Psi_{\frac{n-3}{2}}(\lambda B) \\ &+ \frac{e^{i\lambda(A+B)}}{A^{n-2}B^{n-2}} \lambda^{j-1} \Psi_{\frac{n-3}{2}}(\lambda A)\Psi_{\frac{n-3}{2}}(\lambda B)\Big) \tilde{\Phi}(\lambda)\, d\lambda. \end{aligned} \tag{30}$$

The first term is supported where $\lambda \lesssim \langle A\rangle^{-1}$ and is bounded pointwise by $A^{2-n}\lambda^{n-3+j}$, so its integral is dominated by $A^{2-n}\langle A\rangle^{2-n-j}$. Recalling that $A \approx B$, all other terms are empty unless $A, B \gtrsim 1$.

Note that the next three terms all contain products with both $\Psi_{\frac{n-3}{2}}$ and $\Omega$. Since $A \approx B$, these products are supported in an interval $\lambda \approx A^{-1}$ and have size bounded by 1. Consequently each of the integral terms is bounded by $A^{4-2n-j}$ and is zero for small $A$.

The fifth term plays a very significant role. We can write

$$\lambda^{j-1}\Psi_{\frac{n-3}{2}}(\lambda A)\Psi_{\frac{n-3}{2}}(\lambda B) = A^{n-3}F(\lambda),$$

where $F(\lambda)$ satisfies the conditions of Lemma 4.1 with $\beta = n-4+j$. Consequently,

$$\Big| \int_0^\infty \frac{e^{i\lambda(A-B)}}{A^{n-2}B^{n-2}} \lambda^{j-1} \Psi_{\frac{n-3}{2}}(\lambda A)\Psi_{\frac{n-3}{2}}(\lambda B)\tilde{\Phi}(\lambda\, d\lambda\Big| \lesssim \frac{1}{A^{n-1}\langle A-B\rangle^{n-3+j}},$$

and is also zero for small $A$.

The final term is treated in much the same way, however the phase function $e^{i\lambda(A+B)}$ leads to a stronger bound of $A^{1-n}\langle A+B\rangle^{3-n-j} \approx A^{4-2n-j}$, since $A \gtrsim 1$.

□

The proof of Corollary 2.3 follows by the observation that multiplication by $\lambda$ is essentially the same as taking a radial derivative $\partial_B\big(R_0^+(\lambda^2, B) - R_0^-(\lambda^2, B)\big)$, see (27).

4.2. **Proof of Lemma 2.5.** The main inequality in Lemma 2.5 is as follows.

$$\iint_{\mathbb{R}^{2n}} \frac{|V\phi(z)||V\phi(w)|\,dzdw}{|x-z|^{n-2}\langle|x-z|+|y-w|\rangle\langle|x-z|-|y-w|\rangle^{n-3}} \lesssim \frac{1}{\langle x\rangle^{n-2}\langle|x|+|y|\rangle\langle|x|-|y|\rangle^{n-3}}.$$

The integrals will be set up in spherical coordinates, with radial variables $|x-z|$ and $|y-w|$, so the first step is the following estimate of integrals along shells.

**Lemma 4.2.** *If $N > n-1$, then*

$$\left|\int_{|x-z|=r} \langle z\rangle^{-N} dz\right| \lesssim \begin{cases} r^{n-1}\langle x\rangle^{-N} & r < \frac{1}{2}|x| \\ |x|^{n-1}\langle x\rangle^{1-n}\langle|x|-r\rangle^{n-1-N} & \frac{1}{2}|x| \le r \le 2|x| \\ r^{n-1}\langle r\rangle^{-N} & r > 2|x| \end{cases}$$

*Proof.* The cases $r < \frac{1}{2}|x|$ and $r > 2|x|$ are both trivial, as $|z|$ is comparable to $|x|$ and $|r|$ respectively. In addition, if $r \approx |x| \lesssim 1$, then $\langle z\rangle^{-N} \lesssim 1$ over a sphere of radius approximately $|x|$. This leaves only the case where $r \approx |x|$ and both are relatively large.

We write the integral in spherical coordinates, with $\theta$ representing the angle between $z-x$ and $-x$. Then $|z|^2 = |x|^2 + r^2 - 2|x|r\cos\theta$. We can estimate

$$\int_{|x-z|=r} \langle z\rangle^{-N} dz = C_n \int_0^\pi \frac{r^{n-1}\sin^{n-2}\theta}{\langle(|x|-r)^2 + 2|x|r(1-\cos\theta)\rangle^{N/2}} d\theta.$$

One may replace $\sin\theta$ with $\theta$, and $(1-\cos\theta)$ with $\theta^2$ for the purposes of establishing an upper bound.

On the "cap" where $\theta < ||x|-r|/\sqrt{|x|r}$, the integral may be estimated by

$$\int_0^{\frac{||x|-r|}{\sqrt{|x|r}}} \frac{r^{n-1}\theta^{n-2}}{\langle|x|-r\rangle^N} d\theta \lesssim ||x|-r|^{n-1}\langle|x|-r\rangle^{-N},$$

keeping in mind that $r \approx |x|$. On the remaining interval, we have

$$\int_{\frac{||x|-r|}{\sqrt{|x|r}}}^{\pi} \frac{r^{n-1}\theta^{n-2}}{\langle|x|\theta\rangle^N} d\theta \approx \int_{||x|-r|}^{\pi|x|} \frac{\alpha^{n-2}\,d\alpha}{\langle\alpha\rangle^N} \lesssim \langle|x|-r\rangle^{n-1-N},$$

using the substitution $\alpha = |x|\theta$. □

With this bound in hand, we consider first the $z$ integral of Lemma 2.5.

**Lemma 4.3.** *Let $\beta \ge 1$ and $0 \le \alpha < n-1$. If $N \ge n+\beta$, then for each fixed constant $R \ge 0$, we have the bound*

$$\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-N}}{|x-z|^\alpha\langle|x-z|+R\rangle\langle|x-z|-R\rangle^\beta}\,dz \lesssim \frac{1}{\langle x\rangle^\alpha\langle|x|+R\rangle\langle R-|x|\rangle^\beta}.$$

*Proof.* With $R \geq 0$ a fixed constant, we integrate $\mathbb{R}^n$ in shells of radius $r = |x - z|$.

$$\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-N}}{|x-z|^\alpha \langle |x-z|+R\rangle\langle |x-z|-R\rangle^\beta}\,dz$$
$$= \int_0^\infty \frac{1}{r^\alpha \langle r+R\rangle\langle r-R\rangle^\beta} \int_{|z-x|=r} \langle z\rangle^{-N}\,dz\,dr.$$

By the bounds in Lemma 4.2, we bound the above integral with a sum of three integrals,

$$\int_{\mathbb{R}^n} \frac{\langle z\rangle^{-N}}{|x-z|^\alpha \langle |x-z|+R\rangle\langle |x-z|-R\rangle^\beta}\,dz$$
$$\lesssim \frac{1}{\langle x\rangle^N} \int_0^{|x|/2} \frac{r^{n-1-\alpha}}{\langle r+R\rangle\langle r-R\rangle^\beta}\,dr \tag{31}$$
$$+ \frac{|x|^{n-1-\alpha}}{\langle x\rangle^{n-1}} \int_{|x|/2}^{2|x|} \frac{1}{\langle r+R\rangle\langle r-R\rangle^\beta \langle r-|x|\rangle^{N-n+1}}\,dr \tag{32}$$
$$+ \int_{2|x|}^\infty \frac{r^{n-1-\alpha}}{\langle r+R\rangle\langle r-R\rangle^\beta\langle r\rangle^N}\,dr. \tag{33}$$

We estimate each piece individually. For (31), we consider two cases. First, if $R < \frac{3}{4}|x|$, we cannot use the decay of $\langle r-R\rangle^{-\beta}$ effectively, and instead note that

$$(31) \lesssim \frac{1}{\langle x\rangle^N}\int_0^{|x|/2} \frac{r^{n-1-\alpha}}{\langle r\rangle}\,dr \lesssim \frac{|x|^{n-1-\alpha}}{\langle x\rangle^N} \lesssim \frac{1}{\langle x\rangle^{N-n+\alpha-\beta}\langle |x|+R\rangle\langle R-|x|\rangle^\beta}.$$

The integral bound used here requires $\alpha < n-1$, and we used that $|x| \approx |x| + R$ and $|\,|x| - R| \leq |x|$ in the last step. The desired bound follows, provided that $N \geq n + \beta$.

In the second case, when $R > \frac{3}{4}|x|$,we have $|R - r| > \frac{1}{3}R$ and $r + R \approx R \approx |x| + R$, so that

$$(31) \lesssim \frac{|x|^{n-\alpha}}{\langle x\rangle^N\langle R\rangle^{\beta+1}} \leq \frac{1}{\langle x\rangle^{N-n+\alpha}\langle |x|+R\rangle\langle R-|x|\rangle^\beta}.$$

The last inequality is valid because $|R - |x|\,| \leq R$ in this case.

Now we consider the region on which $r \approx |x|$ in (32). On this region, we have

$$(32) \lesssim \frac{|x|^{n-1-\alpha}}{\langle x\rangle^{n-1}\langle |x|+R\rangle}\int_{r\approx|x|} \frac{1}{\langle r-R\rangle^\beta\langle r-|x|\rangle^{N-n+1}}\,dr.$$

We now extend the integral to $\mathbb{R}$ and apply the simple bound, which asserts that

$$\int_{\mathbb{R}^m} \langle y\rangle^{-\gamma}\langle x-y\rangle^{-\mu}\,dy \lesssim \langle x\rangle^{-\min(\gamma,\mu)}, \tag{34}$$

for all choices $0 < \mu, \gamma$ with $\max(\gamma, \mu) > m$. For the integral under consideration, one chooses $m = 1$, $\gamma = \beta$ and $\mu = N - n + 1$ to obtain

$$(32) \lesssim \frac{1}{\langle x\rangle^\alpha\langle |x|+R\rangle\langle R-|x|\rangle^\beta},$$

provided $N-n+1>\beta$.

Finally, we note consider the final region of integration, (33). We wish to control

$$\int_{2|x|}^{\infty} \frac{r^{n-1-\alpha}}{\langle r+R\rangle\langle r-R\rangle^{\beta}\langle r\rangle^{N}}\, dr.$$

We consider two cases. First, if $R<\frac{3}{2}|x|$, then since $r>2|x|$ this implies that $|R-|x|\,|\leq |r-R|$ and also $r+R\approx r$. Thus, in this case we have

$$\begin{aligned}(33) &\lesssim \frac{1}{\langle R-|x|\rangle^{\beta}}\int_{2|x|}^{\infty}\langle r\rangle^{n-2-\alpha-N}\, dr \lesssim \frac{1}{\langle R-|x|\rangle^{\beta}\langle x\rangle^{N-n+1+\alpha}}\\ &\lesssim \frac{1}{\langle x\rangle^{N-n+\alpha}\langle |x|+R\rangle\langle R-|x|\rangle^{\beta}},\end{aligned}$$

provided $N>n-1-\alpha$ so that the integral converges. We used $|x|+R\leq \frac{5}{2}|x|$ in the last step. The desired bound follows if $N\geq n$.

The second case is when $R>\frac{3}{2}|x|$, in which case we have $|R-|x|\,|\approx R\approx |x|+R$. We break up the region of integration further into three subregions. First, if $2|x|\leq r\leq \frac{R}{2}$, we note that $|R-r|\geq \frac{R}{2}\approx |R-|x|\,|$, permitting the bounds

$$\begin{aligned}\int_{2|x|}^{\frac{R}{2}} \frac{r^{n-1-\alpha}}{\langle r+R\rangle\langle r-R\rangle^{\beta}\langle r\rangle^{N}}\, dr &\lesssim \frac{1}{\langle R\rangle\langle R-|x|\rangle^{\beta}}\int_{2|x|}^{\infty}\frac{r^{n-1-\alpha}}{\langle r\rangle^{N}}\, dr\\ &\lesssim \frac{1}{\langle |x|+R\rangle\langle R-|x|\rangle^{\beta}\langle x\rangle^{N-n+\alpha}}.\end{aligned}$$

The next region we consider is when $\frac{R}{2}\leq r\leq 2R$. Since $r\approx R$, we gain nothing from $\langle R-r\rangle^{-\beta}$ and treat it as a constant. Instead, we note that $|R-|x|\,|\approx R$, and thus we bound with

$$\begin{aligned}\int_{\frac{R}{2}}^{2R} \frac{r^{n-1-\alpha}}{\langle r+R\rangle\langle R-r\rangle^{\beta}\langle r\rangle^{N}}\, dr &\lesssim \frac{R^{n-\alpha}}{\langle R-|x|\rangle^{\beta}\langle R\rangle^{N+1-\beta}}\\ &\lesssim \frac{1}{\langle R-|x|\rangle^{\beta}}\frac{1}{\langle R\rangle^{N-n+1+\alpha-\beta}}.\end{aligned}$$

Then, we note that $|x|\lesssim R$, in order to bound with

$$\frac{1}{\langle x\rangle^{N-n+\alpha-\beta}\langle |x|+R\rangle\langle R-|x|\rangle^{\beta}}.$$

This yields the desired bound, provided $N\geq n+\beta$.

We now consider the last case in which $2R\leq r$. We note that $|R-r|\approx r\approx r+R$ to bound

$$\int_{2R}^{\infty} \frac{r^{n-1-\alpha}}{\langle r+R\rangle\langle R-r\rangle^{\beta}\langle r\rangle^{N}}\, dr \lesssim \int_{2R}^{\infty}\frac{1}{\langle r\rangle^{N-n+\alpha+\beta+2}}\, dr \lesssim \frac{1}{\langle R\rangle^{N-n+1+\alpha+\beta}}.$$

We now note that $|R-|x|\,|\approx R$ and $R>\frac{3}{2}|x|$ to bound with

$$\frac{1}{\langle x\rangle^{N-n+\alpha}\langle |x|+R\rangle\langle R-|x|\rangle^{\beta}}.$$

This yields the desired bound provided $N\geq n$ $\square$

We now conclude with the proof of Lemma 2.5.

*Proof of Lemma 2.5.* Recall that we wish to bound the following,

$$\iint_{\mathbb{R}^{2n}}\frac{|V\phi(z)||V\phi(w)|}{|x-z|^{n-2}\langle |x-z|+|w-y|\rangle\langle |x-z|-|y-w|\rangle^{n-3}}\,dz\,dw.$$

We consider the $z$ integral first. Using Lemma 2.1 and the assumed decay $|V(z)|\lesssim\langle z\rangle^{-(n-1)-}$, it follows that $|V\phi(z)|\lesssim\langle z\rangle^{-N}$ for some $N>2n-3$. Fix the constant $R=|w-y|\geq 0$ and apply Lemma 4.3 (with parameters $\alpha=n-2$ and $\beta=n-3$), to conclude that $K(x,y)$ is bounded by the integral

$$\frac{1}{\langle x\rangle^{n-2}}\int_{\mathbb{R}^n}\frac{\langle w\rangle^{-N}}{\langle |x|+|w-y|\rangle\langle |w-y|-|x|\rangle^{n-3}}\,dw.$$

We again apply Lemma 4.3, this time in $w$ with $R=|x|\geq 0$ and $\alpha=0$, to bound with

$$\frac{1}{\langle x\rangle^{n-2}\langle |x|+|y|\rangle\langle |y|-|x|\rangle^{n-3}},$$

as desired. $\square$

**Lemma 4.4.** *Suppose $|y|>10$ and $0<s\leq 1$. Let $\alpha>0$ and $\beta\geq 1$. If $N\geq n+\beta$, then for each fixed constant $R\geq 0$, we have the bound*

$$\int_{|w|<\frac{|y|}{2}}\frac{\langle w\rangle^{-N}}{|y-sw|^{\alpha}\langle |y-sw|+R\rangle\langle |y-sw|-R\rangle^{\beta}}\,dw\lesssim\frac{1}{\langle y\rangle^{\alpha}\langle |y|+R\rangle\langle R-|y|\rangle^{\beta}}. \tag{35}$$

**Remark 4.5.** *The extra assumptions of large $y$ and relatively small $w$ allow us to remove the upper restriction on the size of $\alpha$. This is very important because we need $\alpha=n-1$ and $\alpha=n$ for the cases of $P_eV1=0$ and $P_eVx=0$ respectively.*

*Proof.* We prove this in a similar manner to Lemma 4.3, taking care to show that the new parameter $s$ is essentially harmless if $|y|$ is large. We begin by decomposing the integral into shells,

$$\begin{aligned}\int_{\mathbb{R}^n}&\frac{\langle w\rangle^{-N}}{|y-sw|^{\alpha}\langle |y-sw|+R\rangle\langle |y-sw|-R\rangle^{\beta}}\,dw\\&=\int_0^{\infty}\frac{1}{|sr|^{\alpha}\langle sr+R\rangle\langle sr-R\rangle^{\beta}}\int_{|y-sw|=sr}\langle w\rangle^{-N}\,dw\,dr\end{aligned}$$

$$= \int_0^\infty \frac{1}{|sr|^\alpha \langle sr+R\rangle \langle sr-R\rangle^\beta} \int_{|w-\frac{y}{s}|=r} \langle w\rangle^{-N}\, dw\, dr.$$

As before, we can use Lemma 4.2 to break this into three pieces. The relevant bound is

$$\Big| \int_{|w-\frac{y}{s}|=r} \langle w\rangle^{-N} dw \Big| \lesssim \begin{cases} r^{n-1}\langle y/s\rangle^{-N} & r < \frac{|y|}{2s} \\ \big(\frac{|y|}{s}\big)^{n-1}\langle \frac{y}{s}\rangle^{1-n}\langle r-\frac{|y|}{s}\rangle^{n-1-N} & \frac{|y|}{2s} \le r \le \frac{2|y|}{s} \\ r^{n-1}\langle r\rangle^{-N} & r > \frac{2|y|}{s} \end{cases}.$$

However we are not integrating over all of $\mathbb{R}^n$. In fact, the ball $|w| < \frac{|y|}{2}$ consists of points where $sr = |y - sw|$ lies between $\frac{1}{2}|y|$ and $\frac{3}{2}|y|$ and is bounded away from zero. Thus only the shell where $r \approx \frac{|y|}{s}$ makes any contribution to the integral. Furthermore, $\frac{|y|}{s} \gtrsim 1$, so the factors $(\frac{|y|}{s})^{n-1}\langle \frac{y}{s}\rangle^{1-n}$ neatly cancel each other.

We use that $sr \approx |y|$ to see that the contribution of the integral can be bounded by

$$\int_{\frac{|y|}{2s}}^{2\frac{|y|}{s}} \frac{1}{|sr|^\alpha \langle sr+R\rangle \langle sr-R\rangle^\beta \langle r-\frac{|y|}{s}\rangle^{N-n+1}}\, dr$$
$$\lesssim \frac{1}{|y|^\alpha \langle |y|+R\rangle} \int_{\frac{|y|}{2s}}^{2\frac{|y|}{s}} \frac{1}{\langle sr-R\rangle^\beta \langle r-\frac{|y|}{s}\rangle^{N-n+1}}\, dr.$$

We make the change of variables $q = rs$, to see

$$\frac{1}{|y|^\alpha \langle |y|+R\rangle} \int_{\frac{|y|}{2}}^{2|y|} \frac{1}{\langle q-R\rangle^\beta \langle \frac{q-|y|}{s}\rangle^{N-n+1}} \frac{dq}{s}.$$

We note that if $||y|-R| \lesssim 1$, then (35) is satisfied so long as the integral above is bounded uniformly in $s$. When $q \approx r$, we replace $\langle q-R\rangle^{-\beta}$ by a constant and observe that remaining expression is a portion of $\int_{\mathbb{R}} \langle r - \frac{|y|}{s}\rangle^{n-1-N}\, dr$, which is integrable and independent of $s$ provided $N > n$.

There are two cases to consider when $||y|-R|| \gtrsim 1$. The first case is when $|q-|y|| < \frac{1}{2}|R-|y||$. Here, we use $|q-R| = |q-|y|+|y|-R| \ge ||y|-R| - |q-|y|| > \frac{1}{2}||y|-R|$. Thus, we bound with

$$\frac{1}{|y|^\alpha \langle |y|+R\rangle \langle |y|-R\rangle^\beta} \int_{\frac{|y|}{2}}^{2|y|} \frac{1}{\langle \frac{q-|y|}{s}\rangle^{N-n+1}} \frac{dq}{s}$$
$$\lesssim \frac{1}{|y|^\alpha \langle |y|+R\rangle \langle |y|-R\rangle^\beta} \int_{\mathbb{R}} \frac{1}{\langle r-\frac{|y|}{s}\rangle^{N-n+1}}\, dr \lesssim \frac{1}{|y|^\alpha \langle |y|+R\rangle \langle |y|-R\rangle^\beta}$$

as desired.

In the second case when $|q-|y|| > \frac{1}{2}|R-|y|| \gtrsim 1$, so that $\frac{q-|y|}{s} \gtrsim 1$. In this case, we bound with

$$\begin{aligned}\frac{1}{|y|^\alpha\langle |y|+R\rangle}&\int_{|q-|y||>\frac12|R-|y||}\frac{dq}{\langle q-R\rangle^\beta\big(\frac{q-|y|}{s}\big)^{N-n+1}s}\\ &\lesssim \frac{s^{N-n}}{|y|^\alpha\langle |y|+R\rangle}\int_{|q-|y||>\frac12|R-|y||}\frac{dq}{\langle q-R\rangle^\beta\langle q-|y|\rangle^{N-n+1}}\\ &\lesssim \frac{s^{N-n}}{|y|^\alpha\langle |y|+R\rangle}\int_{\mathbb{R}}\frac{dq}{\langle q-R\rangle^\beta\langle q-|y|\rangle^{N-n+1}}\lesssim \frac{s^{N-n}}{|y|^\alpha\langle |y|+R\rangle\langle R-|y|\rangle^\beta}.\end{aligned}$$

Where the last inequality follows from (34) provided $N>n-1+\beta$.

□

## 5. Completing the cases $n=6,8,10$

In dimensions $n=8,10$, we still have the task of controlling the contribution of $W_{\log}$, which vanishes only if $P_eV1=0$. From the expansions for $(1+R_0^+(\lambda^2)V)^{-1}$ in Theorem 2.3 of [26], this term takes the form

$$W_{\log}=\frac{1}{\pi i}\int_0^\infty R_0^+(\lambda^2)[V(P_eV)\otimes V(P_eV)](R_0^+(\lambda^2)-R_0^-(\lambda^2))\tilde\Phi(\lambda)\lambda^{j-1}(\log\lambda)^\ell\,d\lambda \tag{36}$$

To control the contribution of these terms, we can adjust the techniques used previously to control the operator $W_{s,2}$. We use the following modification of Lemma 4.1.

**Lemma 5.1.** *Suppose there exists $\beta>-1$ and $M>\beta+1$ such that $|F^{(k)}(\lambda)|\lesssim\lambda^{\beta-k}|\log\lambda|^\ell$ for all $0\le k\le M$. Then given a smooth cutoff function $\tilde\Phi$,*

$$\Big|\int_0^\infty e^{i\rho\lambda}F(\lambda)\tilde\Phi(\lambda)\,d\lambda\Big|\lesssim\langle\rho\rangle^{-\beta-1}\langle\log\langle\rho\rangle\rangle^\ell. \tag{37}$$

*If $F$ is further assumed to be smooth and supported in the annulus $L\lesssim\lambda\lesssim1$ for some $L>\rho^{-1}>0$, then*

$$\Big|\int_0^\infty e^{i\rho\lambda}F(\lambda)\tilde\Phi(\lambda)\,d\lambda\Big|\lesssim\langle\rho\rangle^{-M}L^{\beta+1-M}\langle\log L\rangle^\ell. \tag{38}$$

*Proof.* The proof follows as in the proof of Lemma 4.1, with a few simple modifications. When $0<\lambda<\rho^{-1}$, it follows that

$$\Big|\int_0^{\rho^{-1}}\lambda^\beta(\log\lambda)^\ell\,d\lambda\Big|\lesssim\rho^{-\beta}\Big|\int_0^{\rho^{-1}}(\log\lambda)^\ell\,d\lambda\Big|\lesssim\rho^{-\beta-1}\langle\log\rho\rangle^\ell.$$

On the other hand, if $\lambda>\rho^{-1}$, we note that on the support of $\tilde\Phi$, that $|\log\lambda|\lesssim|\log\rho|$. The second claim follows similarly with $L$ replacing $\rho^{-1}$.

Finally, for $\rho<1$, integrability of $F$ ensures that the left side of (37) is bounded by a constant independent of $\rho$. Inequality (38) is vacuously true for $\rho\ll1$ because then

$L > \rho^{-1} \gg 1$ does not allow $F(\lambda)$ to be non-zero in the support of the integral under the assumptions of the Lemma.

$\square$

Similar to the proof of Lemma 2.2, we can prove

**Lemma 5.2.** *Let $R_0^{\pm}(\lambda^2, A)$ denote the convolution kernel of $R_0^{\pm}(\lambda^2)$ evaluated at a point with $|x| = A$. For each $j \geq 0$,*

$$\int_0^\infty R_0^+(\lambda^2, A)\big(R_0^+ - R_0^-\big)(\lambda^2, B)\lambda^{j-1}(\log\lambda)^\ell \tilde{\Phi}(\lambda)\, d\lambda \tag{39}$$

$$\lesssim \begin{cases} \frac{\langle \log\langle A\rangle\rangle^\ell}{A^{n-2}\langle A\rangle^{n-2+j}} & \text{if } A > 2B \\ \frac{\langle \log\langle B\rangle\rangle^\ell}{A^{n-2}\langle B\rangle^{n-2+j}} & \text{if } B > 2A \\ \frac{\langle \log\langle A-B\rangle\rangle^\ell}{A^{n-2}\langle A\rangle\langle A-B\rangle^{n-3+j}} & \text{if } A \approx B \end{cases}$$

*This can be written more succinctly as*

$$\int_0^\infty R_0^+(\lambda^2, A)(R_0^+ - R_0^-)(\lambda^2, B)\lambda^{j-1}(\log\lambda)^\ell \tilde{\Phi}(\lambda)\, d\lambda \lesssim \frac{\langle \log\langle A-B\rangle\rangle^\ell}{A^{n-2}\langle A+B\rangle\langle A-B\rangle^{n-3+j}}. \tag{40}$$

The proof follows by simply following the proof of Lemma 2.2 using Lemma 5.1 in place of Lemma 4.1. In the regime where $A \approx B$ most of the terms in the decomposition (30) can be bounded by $A^{2-n}\langle A\rangle^{2-n-j}\langle \log\langle A\rangle\rangle^\ell$, except for the fifth term which is bounded instead by $A^{1-n}\langle A-B\rangle^{3-n-j}\langle \log\langle A-B\rangle\rangle^\ell$. Since $A$ and $B$ are positive numbers, $1 \leq \langle A-B\rangle \leq \langle A\rangle$. Moreover, so long as $\ell \leq n-3+j$ the function $G(t) = \frac{\langle \log t\rangle^\ell}{t^{n-3+j}}$ is decreasing for all $t > 1$. The end result is that the upper bound for the fifth term is always an effective upper bound for the entire sum.

When $n = 8, 10$, the form of $W_{\log}$ expressed in [26] consists of a single term with the values $j = n-4$, $\ell = 1$, which certainly satisfies $\ell \leq n-3+j$.

We can then repeat the arguments in Lemma 2.5 with the lazy bound $\langle \log\langle A-B\rangle\rangle \lesssim \langle A-B\rangle^{0+}$. This bounds the kernel of $W_{\log}$ by the quantity

$$\begin{aligned} &\iint_{\mathbb{R}^{2n}} \frac{|V\tilde{\phi}(z)||V\tilde{\phi}(w)|\langle \log\langle |x-z|-|y-w|\rangle\rangle^\ell}{|x-z|^{n-2}\langle |x-z|+|y-w|\rangle\langle |x-z|-|y-w|\rangle^{n-3+j}}\, dzdw \\ &\quad\lesssim \iint_{\mathbb{R}^{2n}} \frac{|V\tilde{\phi}(z)||V\tilde{\phi}(w)|}{|x-z|^{n-2}\langle |x-z|+|y-w|\rangle\langle |x-z|-|y-w|\rangle^{n-3+j-}}\, dzdw \\ &\quad\lesssim \frac{1}{\langle x\rangle^{n-2}\langle |x|+|y|\rangle\langle |x|-|y|\rangle^{n-3+j-}}. \end{aligned}$$

where $\tilde{\phi}$ satisfies the same decay properties as an eigenfunction $\phi$. Since $j = n-4 > 2$, this is sufficient to make $W_{\log}$ an admissible kernel that is bounded on $L^p(\mathbb{R}^n)$ for all $1 \leq p \leq \infty$.

If $n = 6$ the structure of $W_{\log}$ is more complicated. Assuming that $|V(x)| \leq C\langle x\rangle^{-\beta}$ for $\beta > 10$, Theorem 2.3 of [26] provides the expression

$$W_{\log} = \sum_{j,\ell=1}^{2} \sum_{a,b=1}^{2d} \int_0^\infty R_0^+(\lambda^2)[\varphi_a \otimes \psi_b](R_0^+(\lambda^2) - R_0^-(\lambda^2))\tilde{\Phi}(\lambda)\lambda^{2j-1}(\log\lambda)^\ell\, d\lambda, \tag{41}$$

where $d$ is the (finite) dimension of the zero energy eigenspace, and each $\varphi_a, \psi_b$ belongs to $\langle x\rangle^{-\beta+3+}H^2(\mathbb{R}^6)$. Dependence of $\varphi_a$ and $\psi_b$ on the parameters $j$ and $\ell$ is suppressed in the notation above. Each term in the sum yields an integral kernel that is bounded by

$$\begin{aligned}\iint_{\mathbb{R}^{2n}} &\frac{|\varphi_a(z)||\psi_b(w)|\langle\log\langle|x-z|-|y-w|\rangle\rangle^\ell}{|x-z|^4\langle|x-z|+|y-w|\rangle\langle|x-z|-|y-w|\rangle^{3+2j}}\, dz\, dw \\ &\lesssim \iint_{\mathbb{R}^{2n}} \frac{|\varphi_a(z)||\psi_b(w)|}{|x-z|^4\langle|x-z|+|y-w|\rangle\langle|x-z|-|y-w|\rangle^{3+2j-}}\, dz\, dw.\end{aligned}$$

It is not possible to invoke Lemma 4.3 directly because we lack a pointwise bound for functions $\varphi_a$, $\psi_b$. However they do belong to the space $\langle x\rangle^{-\beta+3+}L^6(\mathbb{R}^6)$ by Sobolev embedding. Then by Hölder's inequality we may write

$$\begin{aligned}\int_{\mathbb{R}^6} &\frac{|\varphi_a(z)|}{|x-z|^4\langle|x-z|+R\rangle\langle|x-z|-R\rangle^{3+2j-}}\, dz \\ &\leq \|\langle z\rangle^{\beta-3-}\varphi_a\|_6 \ \left\|\frac{1}{\langle z\rangle^{1/6}\langle|x-z|+R\rangle^{1/6}}\right\|_\infty \\ &\quad \times \left(\int_{\mathbb{R}^6} \frac{\langle z\rangle^{-\frac{6}{5}(\beta-\frac{1}{6}-3-)}}{(|x-z|^4\langle|x-z|-R\rangle^{3+2j-})^{6/5}\langle|x-z|+R\rangle}\, dz\right)^{5/6} \\ &\lesssim \|\langle z\rangle^{\beta-3-}\varphi_a\|_6 \ \langle|x|+R\rangle^{-\frac{1}{6}}\left(\frac{1}{\langle x\rangle^4\langle|x|+R\rangle^{5/6}\langle|x|-R\rangle^{3+2j-}}\right) \\ &\lesssim \frac{1}{\langle x\rangle^4\langle|x|+R\rangle\langle|x|-R\rangle^{3+2j-}}.\end{aligned}$$

The $L^\infty$ bound is observed via the inequality $\langle z\rangle\langle|x-z|+R\rangle \gtrsim \langle|z|+|x-z|+R\rangle \geq \langle|x|+R\rangle$. In order to apply Lemma 4.3 to the last integral, we need the singularity $|x-z|^{-24/5}$ to have an exponent less than $6-1$, and for $\frac{6}{5}(\beta - \frac{19}{6}-) \geq 6 + \frac{6}{5}(3+2j-)$. The first condition is true, and the second is satisfied provided $\beta \geq 11 + \frac{1}{6} + 2j$. Allowing $\beta > 16$ suffices in all cases.

The integral involving $\psi_b(w)$ is handled in an identical manner. After summing over $a$, $b$, and $\ell$, it follows that the kernel for $W_{\log}$ is bounded by

$$\sum_{j=1}^{2} \frac{1}{\langle x\rangle^4\langle|x|+|y|\rangle\langle|x|-|y|\rangle^{3+2j-}}$$

If $j = 2$ this is an admissible kernel whose operator is bounded on $L^p(\mathbb{R}^6)$ for all $1 \le p \le \infty$. If $j = 1$, the bound of $\frac{1}{\langle x\rangle^4 \langle y\rangle^{-6-}}$ for large $y$ just fails to be integrable. One can follow the proof of Proposition 2.6 to determine that this integral operator is still bounded on $L^p(\mathbb{R}^6)$ for all $1 \le p < \infty$, missing only the $p = \infty$ endpoint.

We believe that careful analysis of the operators $D_{jk}^{(i)}$ derived in [9] would show that $W_{\log}$ vanishes when $P_e V 1 = 0$ in dimension six, just as it does when $n = 8, 10$. Even without this claim, however, the bound for $W_{\log}$ given here suffices to complete the proof of Theorem 1.1 in all of its cases.

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